Review and Application of Prestressed Shell Theories to Blood Vessel Wave Propagation

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Theme

NUMEROUS theories for the analysis of thin-walled shells have been developed primarily for the solution of stress and stability problems that arise in the domain of solid mechanics. Some of the existing theories are reviewed to ascertain their influence on the computation of phase velocities in fluid filled cylinders representing certain aspects of the behavior of arteries and veins in vivo.

Content

In many situations involving the dynamic behavior of blood vessels it is essential to account for the significant prestresses to which the vessels are subjected. Also, it is clearly established that the constitutive law of any tissue is very complex, e.g., Fung.[1] In addition, King and Lawton,[2] Anliker et al.,[3] Anliker et al.[4] have demonstrated that the elastic response of the vessel to the natural cardiac pulse can be significantly nonlinear. However, for a number of investigations such as those devoted to determination of the wall material properties, we are primarily concerned with the behavior of the vessel as it responds in a small neighborhood of a quasi-static prestressed state. In the experiments the wall material properties are deduced by measuring the amplitude and phase differences for mechanically induced high-frequency waves propagating down the vessel, as described in Ref. 3, for example. For such cases the perturbation strains of interest are small and it is reasonable, at least as a first approximation, to treat the vessel wall material as perfectly elastic and consider the prestressed state as given.

In reviewing the well-known literature on prestressed shells, e.g., Refs. 5-10, it is immediately obvious, as in many other fields of shell analysis, that the equations used by the various authors do not agree in all respects. It is appropriate to study whether such differences have any significant influence on the phase velocities in simplified models of veins and arteries. Another subject of considerable interest is the fact that unlike metal structures, the inplane strains induced by the prestress in the primary arteries are not small compared to unity.

Equations for Cylindrical Shells under Initial Stress when the Strains are Small Compared to Unity

Let us restrict our attention to those sets of equations in which the strains are considered small compared to unity, while retaining all terms involving the prestress resultants. Since the resulting linear displacement equilibrium equations can be derived from a variational principle it is known (see e.g., Ref. 6) that their operators are self adjoint. For two dimensional systems, the adjoint operator of $\partial\{a(\alpha,\beta)u(\alpha,\beta)\}/\partial\alpha$, where $a(\alpha,\beta)$ is a coefficient of the independent variables only, is by definition $(-1)a(\alpha,\beta)\partial u(\alpha,\beta)/\partial\alpha$ and similarly for higher order derivatives. While this relationship is satisfied in most published works it is not obeyed, for example in Bolotin's book[9] where the error is made of deriving nonlinear membrane equations for the prestress terms by multiplying the Lame coefficients in the linear equations by unity plus the relevant strain to obtain the nonlinear equations.

Thin Circular Cylindrical Shell Analyses Allowing for Membrane Strains that are Nonlinear in the Inplane Displacements

Since self adjointness is evidenced by symmetry of the differential operators in Cartesian coordinates we have reason to be curious about this apparent conflict with Biot's[11] and Pflüger's[12] demonstrations that the equations of elasticity for incremental deformations are nonsymmetric in Cartesian coordinates. The clue to this paradox comes from the fact that the majority of authors concerned primarily with the applications of shell theory, neglect, in the strain displacement relations, all nonlinear terms involving the tangential displacements u and v. Hence when the potential energy contains displacement terms to powers higher than the second, whether the final equations from which the eigenmatrix is formed are self adjoint or not, depends on the assumptions made to linearize the Euler equations of the variational method. These resulting equilibrium equations[6,7] consist of terms of two types, namely stress resultants and their derivatives plus stress resultants times midsurface rotations, and those consisting of stress resultants times midsurface strains. These latter terms result from a variational formulation only when the nonlinear terms in the inplane displacements are retained in the strain-displacement relations.

A simple situation, of frequent interest, particularly in the field of biomechanics is when uniform axial stretch and internal pressure constitute the prestressed state. If we now assume an isotropic shell in which the Young's modulus is independent of the prestressed state the perturbation equations assume a relatively elegant form.

Now we may write the equations for perturbations about a prestressed state defined by the constant strains in differential operator form as

$$[L_{ij}]\begin{vmatrix} u \\ v \\ w \end{vmatrix} + \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = 0$$

In which the off diagonal operators are

$$L_{21} = \frac{(1+\nu)}{2R}(1+l_\varphi^o)\frac{\partial^2}{\partial x\partial\varphi}, \quad L_{12} = \frac{(1+\nu)}{2R}(1+l_x^o)\frac{\partial^2}{\partial x\partial\varphi}$$

$$L_{31} = (1+l_\varphi^o)(\nu/R)\,\partial/\partial x, \qquad L_{13} = -(1+l_x^o)(\nu/R)\,\partial/\partial x$$

$$L_{32} = (1/R^2)(1+3l_\varphi^o+2\nu\, l_x^o)\,\partial/\partial\varphi - d^2\{(1/R^2)\,\partial^3/\partial\varphi^3 + (2-\nu)\,\partial^3/\partial x^2\partial\varphi\}$$

$$L_{23} = -(1/R^2)(1+3l_\varphi^o+2\nu\, l_x^o)\,\partial/\partial\varphi + d^2\{(1/R^2)\,\partial^3/\partial\varphi^3 + (2-\nu)\,\partial^3/\partial x^2\partial\varphi\}$$

l_x^o and l_φ^o are the initial membrane strains. It is clear that these operators are self adjoint for large strains only when $l_x^o = l_\varphi^o$,

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i.e., when the prestresss state is hydrostatic, substantiating completely Biot's and Pflüger's conclusion. The usual manner in which the influence of the prestressed state is illustrated, see for example Refs. 11 and 12, is to show that the nonsymmetry in Cartesian coordinates occurs in the stress strain relations and the degree of asymmetry is of the order of the initial stress as divided by the elastic modulus. In our derivation, the classic symmetric stress strain law was assumed and the degree of nonself-adjointness was also shown to be of the order of the initial strain. In virtually all metallic structures the initial membrane strains are indeed small compared to unity. However, in blood flow problems they can be as high or higher than 0.6 and neglecting them is less justified.

For axisymmetric waves and zero circumferential prestress, the predictions based on Refs. 6, 8 and 10 agree very closely with each other for the pressure and axial waves. The apparent lack of agreement with Ref. 14 is solely due to a different assumption concerning the prestress application. However, for nonzero circumferential prestress there are some significant variations among the results. They are most easily discussed by considering the uncoupled torsion waves separately from the pressure and axial waves, whose equations are coupled.

Torsion Waves

A somewhat surprising result is that unlike other shell equations, those of Ref. 10 and Ref. 6 plus a number of other authors predict a cutoff frequency for the type II or torsion waves whenever the shell is subjected to transmural pressures. The cause for this can be seen by noting that the equation for axisymmetric torsion waves becomes uncoupled from the other two. In Ref. 14 we have the characteristic equation

$$c^2 = \left(\frac{\sigma}{k}\right)^2 = \left[\frac{(1-\nu)(1+3d^2)/2 + N_x^o + \nu N_\varphi^o}{\rho}\right]$$

where c is the nondimensional wave speed, $d^2 = h^2/12\,R^2$ and the other symbols have their usual meaning. According to this equation the torsion waves are nondispersive and have no cutoff frequency below which they do not propagate. This is in contradiction with the other two theories mentioned which yield

$$c^2 = \left(\frac{\sigma}{k}\right)^2 = \frac{(1-\nu)\dfrac{(1+9d^2/4)}{2} + \dfrac{(N_x^o + N_\varphi^o)}{2}}{\rho} + \frac{N_\varphi^o}{\rho k^2}$$

In both of these cases it is the last term $(N_\varphi^o/\rho k^2)$ that introduces both the dispersion, and the cutoff frequency for non-zero transmural pressure. Therefore $\sigma = (N_\varphi^o/\rho)^{1/2}$ is the cutoff frequency below which waves do not propagate.

The erroneous cutoff frequency arises, see Ref. 13, when certain terms are omitted in the derivation of the hydrostatic pressure force induced by the perturbation rotations.

Pressure and Axial Waves

While the differences in strain displacement relations in the various formulations do cause differences in the phase velocities of Pressure and Axial waves, they are not truly significant and in many situations the theories are in virtual agreement.

Table 1 Typical modal coefficients for the Anliker-Maxwell[14] and Washizu[6] theory for $N_x^0 = 0.4$, $N_\varphi^0 = 0$, and $k = 5.0$

	Shell theory modal displacement								
	Anliker-Maxwell[14]			Washizu[6] Basic system			Washizu[6] adjoint system		
Wave type	Axial	Torsion	Radial	Axial	Torsion	Radial	Axial	Torsion	Radial
I	−0.044	0	1.0	−0.0469	0	1.0	−0.0335	0	1.0
II	0	1.0	0	0	1.0	0	0	1.0	0
III	1.0	0	0.044	1.0	0	0.0335	1.0	0	0.0469

Consideration of Nonself-Adjointness of Washizu's Equations

Attention has been drawn to the fact that the "moderate strain" form of Washizu's[6] accounting for nonlinear inplane displacements, yields displacement equilibrium equations which are nonself-adjoint and thus we know from well established theory, e.g. Ref. 15, that the eigenvalues of the basic system and its adjoint are identical, but that the eigenvectors of the two systems are different and form a biorthogonal set. The modal amplitude coefficients are indeed slightly different and Table 1 shows the typical results.

These results illustrate the theory by showing that for the self-adjoint operators of Ref. 14, the modes are orthogonal, while those of Ref. 6 for both the basic and adjoint equations are not orthogonal, but they are biorthogonal. However, the fact that the modes predicted by Ref. 6 are not quite orthogonal is of negligible importance in the study of high-frequency waves induced mechanically in the cardiovascular system.

The phase velocities predicted by the small strain theories of Refs. 10 and 13, and the theory of Ref. 5 are considered to be correct. All three wave speeds are essentially independent of internal hydrostatic pressure if the cylinder walls are not axially constrained. For future studies of this type on shells of general geometry, those of Budiansky 10 are recommended.

The nonself-adjointness of the moderate membrane strain theory of Washizu[6] is shown to occur due to the inclusion of nonlinear inplane displacements in the strain displacement relations. The modes obtained from the resulting eigenmatrix are very similar to those of the adjoint eigenmatrix. Thus, the nonself-adjointness character of the equations is of negligible importance for the physiological problems of wave propagation in the arterial and venous systems.

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